Elliptical Orbit with Variable Angular Momentum

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Introduction

STUDENTS of classical mechanics are usually taught that for general initial conditions, orbits that remain closed for large deviation from circularity are those having a force law $g(r) = -k/r^2$ (inverse-square law, k constant) or $g(r) = -kr$ (Hooke's law). This is known as Bertrand's theorem, a clear discussion of which is given in Goldstein.[1] In such cases no orbit precession is observed.[2] Recent work by Gorringe and Leach[3,4] (see also references therein) has shown that in contrast to the classical result of Bertrand, closed conic sections exist for a variety of problems with variable angular momentum. The present study extends these previous studies[3,4] by establishing general conditions for planar motion given by Eq. (1) to describe a constant elliptical orbit. Relation with Kepler's laws will be discussed, and some applications will be given in the last section.

Planar Elliptical Motion

The procedure followed is similar to that given by Gorringe and Leach[3] or Mavraganis.[5] The starting point is the differential equation

$$\ddot{r} + f\dot{r} + gr = 0 \tag{1}$$

where the dot indicates differentiation with respect to time. The dependence of f and g on the radial variable r or the angular variable θ (see Fig. 1) and on the derivatives of these variables will become clear later on [see Eqs. (4) and (10)]. By taking the vector product of $\boldsymbol{r}$ with Eq. (1) and by noting that the angular momentum vector $\boldsymbol{\ell} = \boldsymbol{r} \times \dot{\boldsymbol{r}}$ (amplitude $\ell = r^2\dot{\theta}$), one gets

$$\dot{\boldsymbol{\ell}} + f\boldsymbol{\ell} = \boldsymbol{0} \tag{2}$$

By taking the vector product of $\boldsymbol{\ell}$ with Eq. (2), one gets

$$\dot{\boldsymbol{\ell}} \times \boldsymbol{\ell} = 0 \tag{3}$$

For planar motion, the vector $\dot{\boldsymbol{\ell}} = \dot{\ell}\hat{\boldsymbol{\ell}}$ is in the direction of the unit vector $\hat{\boldsymbol{\ell}}$ perpendicular to the plane of motion. From Eq. (2), one gets

$$f = -\frac{\dot{\ell}}{\ell} = -\left(\frac{2\dot{r}}{r} + \frac{\ddot{\theta}}{\dot{\theta}}\right) \tag{4}$$

From Eqs. (1) and (2) one can derive the following vectorial equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{\boldsymbol{r}}}{\ell}\right) = -\frac{g}{\ell}\boldsymbol{r} \tag{5}$$

Equation (5) describes a planar motion with variable angular momentum ℓ, it is equivalent to the two following equations in Cartesian coordinates:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{x}}{\ell}\right) = -\frac{g}{\ell}x \tag{6a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{y}}{\ell}\right) = -\frac{g}{\ell}y \tag{6b}$$

with $x = r\cos\theta$ and $y = r\sin\theta$ (see Fig. 1).

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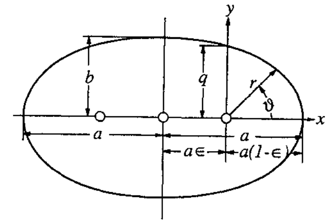

Fig. 1 General arrangement of an ellipse, with equation $r = q - \epsilon x$, a = semi-major axis, b = semi-minor axis, $q = a(1 - \epsilon^2)$ = semi-latus rectum, ϵ = eccentricity. The origin of the coordinates is at the right focus of the ellipse.

From Eq. (5) one can easily derive the differential equation for the radial variable r,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{r}}{\ell}\right) = -\frac{g}{\ell}r + \frac{\ell}{r^3} \tag{7a}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{r}}{\ell}\right) = -\frac{g}{\ell}r + \frac{\dot{\theta}}{r} \tag{7b}$$

On the other hand, the equation of a conic section with constant eccentricity ϵ and constant semilatus rectum q can be written in the form[6]

$$r = q - \epsilon x \tag{8}$$

The origin in this case is at a focus of the ellipse.

By differentiating and then by dividing by ℓ, one gets

$$\frac{\dot{r}}{\ell} = -\epsilon\frac{\dot{x}}{\ell} = \frac{\epsilon}{q}\sin\theta \tag{9a}$$

By multiplying Eq. (9a) by $\dot{\theta}$ and with $\ell = r^2\dot{\theta}$, the left-hand side reduces to $\dot{r}/r^2$ and Eq. (9a) can be integrated to give

$$\frac{r\dot{\theta}}{\ell} = \frac{1}{r} = \frac{1}{q} + \frac{\epsilon}{q}\cos\theta \tag{9b}$$

Equations (9) for the radial velocity $\dot{r}$ and angular velocity $r\dot{\theta}$ of a constant elliptical orbit are derived under the assumption of a variable angular momentum ℓ. It is assumed evidently that the orbital plane is fixed with respect to an inertial frame of reference.

Force Field and Kepler's Laws

From Eqs. (9) and (7b) one can derive

$$\frac{g}{\ell}r = \frac{\dot{\theta}}{q} \tag{10}$$

or

$$g = \frac{r\dot{\theta}^2}{q} = \frac{\ell^2}{qr^3} \tag{11}$$

Equation (11) shows the condition on g in order to have a motion along a constant ellipse described by Eqs. (8) and (9), g is proportional to the centripetal force passing through a focus of the ellipse.

Equation (10) with Eq. (5) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{\boldsymbol{r}}}{\ell}\right) = -\frac{\dot{\theta}}{q}\boldsymbol{i}_r \tag{12}$$

where $\boldsymbol{i}_r$ is a unit vector in the radial direction. Equation (10) and Eq. (7b) give

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{r}}{\ell}\right) = \dot{\theta}\left(-\frac{1}{q} + \frac{1}{r}\right) \tag{13}$$

By noting that $(d/dt)(\dot r/\ell) = -\dot\theta\,(d^2/d\theta^2)u$, where $u = 1/r$, Eq. (13) becomes

$$\frac{d^2u}{d\theta^2} + u = \frac{1}{q} \tag{14}$$

The solution of Eq. (14) is nothing else but Eq. (9b). Note, however, that unlike what is usually done,[5–9] Eq. (14) and its solution Eq. (9b) do not assume that the angular momentum ℓ is constant.

Equation (12) can be written in the traditional way giving the radial and the transverse components of the force

$$\ddot r - r\dot\theta^2 = -\frac{r^2\dot\theta^2}{q} + \frac{\dot\ell}{\ell}\dot r \tag{15a}$$

$$r\ddot\theta + 2\dot r\dot\theta = \frac{\dot\ell}{\ell} r\dot\theta \tag{15b}$$

Note from Eqs. (15) that the force field is noncentral and that the two components of the nonconservative force $(\dot\ell/\ell)\dot r$ and $(\dot\ell/\ell)r\dot\theta$ correspond to the term $f\dot{\mathbf r}$ in Eq. (1). By multiplying Eq. (15a) by $\dot r$ and by direct manipulation of Eq. (15b), one can again write Eqs. (15a) and (15b) in the form

$$\frac{1}{2}\frac{d}{dt}\left(\frac{\dot r}{\ell}\right)^2 - \frac{\dot r}{r^3} = \frac{1}{q}\frac{d}{dt}\left(\frac{1}{r}\right) \tag{16a}$$

$$\frac{1}{r}\frac{d}{dt}(r^2\dot\theta) = \frac{\dot\ell}{r} \tag{16b}$$

Equation (16b) is an identity, it is valid for any value of $\dot\ell$, and the case where $\dot\ell = 2(d^2A/dt^2) = 0$ is a special case; A is the sector area of the ellipse. Kepler's second law, which states that (dA/dt) = const appears as a special case of Eq. (16b) when $\dot\ell = 0$. By integrating Eq. (16a) and by noting that $(r\dot\theta/\ell)^2 = 1/r^2$, one gets the energylike equation

$$\frac{1}{2}\dot s^2 - \frac{\ell^2}{qr} = -\frac{\ell^2}{2qa} \tag{17a}$$

where a is the semimajor axis of the ellipse, $q = a(1-\epsilon^2)$ and s is the arc length, and $\dot s$ is the magnitude of the velocity.

By putting $g = K/r^3$, one gets from Eq. (11)

$$K = \frac{1}{q}\ell^2 \tag{17b}$$

and Eq. (17a) takes the following form well known for Newtonian (or Coulombian) attraction

$$\frac{1}{2}\dot s^2 - \frac{K}{r} = -\frac{K}{2a} \tag{17c}$$

Note, however, from Eq. (17b) that $K \equiv K(r,\dot\theta)$ is not constant. The sector area A of the ellipse is given by

$$2A = \int_0^t \ell\, dt \tag{18a}$$

$$\frac{2A}{\sqrt{q}} = \int_0^t \sqrt{K}\, dt \tag{18b}$$

By writing for the area of the ellipse $A_e = \pi a^2\sqrt{1-\epsilon^2}$; Kepler's third law appears as a special case of Eq. (18b) when K (and ℓ) is constant.

Consequently by writing T = period, one has 1) Kepler's second law (dA/dt) = const and 2) Kepler's third law $4\pi^2a^3 = KT^2$, (K = const) and both appear as a special case of Eqs. (18) when $\dot\ell = 0$ in Eqs. (15) and (16).

Since $\dot\ell = 0$ implies no friction force in Eq. (1), Kepler's laws appear as an ideal case solution with zero friction, since in actual physical situations one can expect to have a friction term $f = -(\dot\ell/\ell) \neq 0$.

Note that by integrating over a whole period, one has

$$2A_e = T\ell_c = \int_0^T \ell\, dt \tag{19a}$$

$$\frac{2A_e}{\sqrt{q}} = T\sqrt{K_c} = \int_0^T \sqrt{K}\, dt \tag{19b}$$

where ℓ_c and $\sqrt{K_c}$ are values obtained by applying the mean value theorem. The elliptical motion with constant ℓ_c and K_c is described by the same equations already derived, with the coordinates (r_c, θ_c) replacing (r, θ). One has, for instance, $1/r_c = (1/q) + (\epsilon/q)\cos\theta_c$, and the relation between (r,θ) and (r_c,θ_c) along the same elliptical orbit is given by

$$\frac{1}{r} - \frac{1}{r_c} = \frac{\epsilon}{q}(\cos\theta - \cos\theta_c) \tag{20}$$

Applications

The purpose of this section is to show some potential applications of this study.

1) By taking $f = -(\dot\ell/\ell) = -1/2(\alpha+3)\dot r/r$, one gets $\ell = kr^{(\alpha+3)/2}$. Equation (11) shows that $g = (k^2/q)r^\alpha = \mu r^\alpha (\mu, k =$ const). These values for f and g are those used in Eq. (2.1) of Gorringe and Leach.[4] But whereas there is no apparent reason why these values are used in Ref. 4, Eq. (11) gives a direct explanation.

2) In Mittleman and Jezewski,[10] $f = (-\dot\ell/\ell) = \alpha/r^2$ which gives $\ell = r^2\dot\theta = h_0 - \alpha\theta$. By Eq. (11) $g = [(h_0-\alpha\theta)^2/q]1/r^3$, and the solution is as outlined in the present study by replacing ℓ by $h_0 - \alpha\theta$. However, because Mittleman and Jezewski[10] use $g \approx \mu/r^3$ (g is γ in their notation), μ = const, the solution they obtain is not a closed orbit (see their Figs. 1 and 2), because the condition of Eq. (11) is not satisfied. If one takes $\ell = h_0 - \alpha\theta$ with $0 \le \theta \le 2\pi$, the solution outlined in the present study is periodic and simpler in form than that of Ref. 10.

Also by writing

$$g = \frac{(h-\alpha\theta)^2}{q}\frac{1}{r^3} = \frac{h^2}{qr^3} - \frac{2h\alpha\theta}{qr^3} + \frac{\alpha^2\theta^2}{qr^3} \tag{21}$$

and by taking $g \approx h^2/qr^3 = \mu/r^3$, one can estimate the correction term $-(2h\alpha\theta/qr^3) + (\alpha^2\theta^2/qr^3)$ needed in the expression of g in order to obtain a closed orbit.

3) An interesting application that deserves further study is the problem of the secular acceleration of the moon's longitude which is modeled as a Keplerian problem with time-dependent force factor K but with constant angular momentum. An interesting review of this problem is given by Deprit.[11]

Conclusion

Some interesting results appear from the present study are as follows.

1) Motion along a constant elliptical orbit with noncentral force and variable angular momentum described by Eq. (1) is possible, a result that is already known,[3,4] but is derived in the present study in a general and simple way without making use of the Lenz or Hamilton vectors.

2) Kepler's second and third laws appear intimately related to Eqs. (15) or Eqs. (16) by putting $\dot\ell = 2(d^2A/dt^2) = 0$. Since motion in nature usually includes friction, the inclusion of the term $\dot\ell \neq 0$ in Eqs. (15) and (16) is a problem that deserves special attention in future applications.

3) Since Eq. (11) introduced in this study gives the condition to have an elliptic orbit described by Eq. (1), knowledge of Eq. (11) can allow better evaluation of possible perturbation forces for the case of slight deviation from elliptical motion.

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Autonomous Ring Formation for a Planar Constellation of Satellites

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Introduction

MULTIPLE-SATELLITE rings have been considered recently for a variety of mission applications. These applications center on global point-to-point communications, such as Teledesic and Iridium.[1] The formation and stationkeeping of such large constellations pose new and interesting problems in orbital dynamics.

To ensure global coverage, the intersatellite spacing must be maintained. Small altitude errors will, if uncorrected, result in phasing drifts and clustering of the satellites. There is a requirement, therefore, to ensure that uniform intersatellite spacing is maintained. To control the dynamics of each satellite individually from a ground station would be both complex and expensive. Therefore, an autonomous system is preferable, resulting in lower operational costs and greater operational flexibility.

Such autonomous stationkeeping has been developed for single satellite platforms.[2] However, with multiple satellite rings the system to be controlled must be considered as the entire, collective ensemble of satellites. As such, the orbit control problem is significantly more complex as a large number of individual satellites must be controlled simultaneously.

In this Note a novel, autonomous ring formation and stationkeeping method is considered. Using simple analytic commands, a "loose" ring of satellites can be formed into a perfect ring with uniform intersatellite spacing. The method has analogies with emergent behavior seen in recent studies of nonlinear systems. That is, a set of simple rules is used to generate complex, emergent behavior that is not designed into the system. For this problem analogies may be drawn with the formation of crystal lattices through the minimization of free energy in the system. The satellite ring formation problem is seen to be somewhat similar.

System Dynamics

The dynamics of a system of N satellites will be considered in orbit around a point mass Earth, Fig. 1. The dynamics of the system may then be represented by a system of $2N$ equations of motion, viz.,

$$\left.\begin{aligned} \ddot{r}_i - r_i\dot{\theta}_i^2 &= -\frac{\mu}{r_i^2} + a_{ri} \\ r_i\ddot{\theta}_i + 2\dot{r}_i\dot{\theta}_i &= a_{\theta i} \end{aligned}\right\} \qquad (i = 1, \ldots, N) \tag{1}$$

where a_{ri} and $a_{\theta i}$ are radial and transverse low-thrust control accelerations assumed to be available from onboard thrusters. It is clear that in the open-loop case this system of equations possesses a particular desired solution (r_i^*, θ_i^*) given by

$$\left.\begin{aligned} r_i^* &= \tilde{r} \\ \theta_i^* &= \tilde{\omega}t + 2i\chi \end{aligned}\right\} \qquad (i = 1, \ldots, N) \tag{2}$$

where $\tilde{r}$ is the operational radius of the ring, χ is the vertex half-angle of the N polygon defined by the ring, and $\tilde{\omega}$ is the Keplerian angular velocity at the operational radius. This solution represents the nominal ring with perfect intersatellite spacing.

To investigate ring formation and stationkeeping the $2N$ equations of motion will be linearized relative to the nominal ring. This may be achieved by defining new variables,

$$\begin{aligned} \phi_i &= \theta_i - \tilde{\omega}t \\ l_i &= r_i - \tilde{r} \end{aligned} \tag{3}$$

where the subscript is now understood to run from 1 to N. The linearized system of equations may now be written in the new variables as

$$\begin{aligned} \ddot{l}_i - 2\tilde{\omega}\tilde{r}\dot{\phi}_i - 3\tilde{\omega}^2 l_i &= a_{li} \\ \tilde{r}\ddot{\phi}_i + 2\tilde{\omega}\dot{l}_i &= a_{\phi i} \end{aligned} \tag{4}$$

In linearizing it has been assumed that $l_i/\tilde{r} \ll 1$ and that $\dot{\phi}_i \ll 1$ but not that ϕ itself is small. This set of linear equations may now be used to generate the controls required for ring formation.

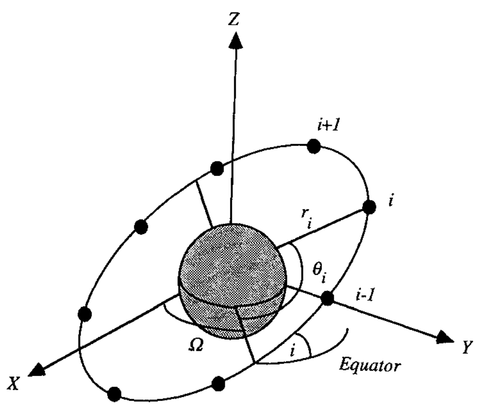

Fig. 1 Schematic geometry of an N-satellite ring.

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